# ON MONADS IN SATURATED ENLARGEMENTS

BY

STEPHEN H. HECHLER

ABSTRACT

We show that under a stronger hypothesis, a theorem of Luxemburg can be sharpened.

Suppose $\mathscr{M} = \mathscr{M}(X, M)$ is the full $L$-structure over $X$, $^*\mathscr{M}$ is a higher order model of $\mathscr{M}$, and $A$ is an entity of type $(\sigma)$ of $^*\mathscr{M}$ (i.e., an *internal* set of entities of type $(\sigma)$). In [1] (from which all our notation is taken) Luxemburg asks about the relationship between the discrete monads $\mu_d(A)$ and $\mu_d(\{A\})$ and proves (Theorem 2.6.2) that if $^*\mathscr{M}$ is an enlargement, then $\mu_d(A) = \mu_d(\cup\, \mu_d(\{A\}))$. We prove:

THEOREM. *Under the above conditions, if $^*\mathscr{M}$ is an enlargement and either $A$ is a singleton or $^*\mathscr{M}$ is saturated, then $\mu_d(A) = \cup\, \mu_d(\{A\})$.*

PROOF. By Luxemburg's theorem it is sufficient to prove that $\mu_d(A) \subseteq \cup\, \mu_d(\{A\})$. We first note that if $R \in M$ and $A \in {}^*R$, then $\mu_d(A) \subseteq \cup\, {}^*R$. This follows easily from the fact that $A \in {}^*R$ implies $A \subseteq \cup\, {}^*R = {}^*(\cup R)$.

Now choose any $a \in \mu_d(A)$. From the above, we see that for each $R \in M$ such that $A \in {}^*R$, we can find an entity $B_R$ such that $a \in B_R \in {}^*R$. We look at two cases.

*Case* 1. $A$ is a singleton. Then for each $R$, look at the set $S = \{r \in R : r \text{ is a singleton}\}$. Since $S$ contains all the singletons of $R$, $^*S$ must contain all the singletons of $^*R$. In particular $A \in {}^*S$. Thus we may find a set $B_S$ such that $a \in B_S \in {}^*S \subseteq {}^*R$. But $^*S$ contains only singletons so $a \in B_S \in {}^*S$ implies $B_S = \{a\}$. Thus we have shown that for each $R \in M$ such that $A \in {}^*R$, we have $\{a\} \in {}^*R$. This in turn tells us that $\{a\} \in \mu_d(\{A\})$ and therefore $a \in \cup\, \mu_d(\{A\})$.

*Case* 2. $^*\mathscr{M}$ is saturated. Let $b$ be the internal relation defined by $\Phi_3(b, D, B)$

Received November 10, 1971.

$\leftrightarrow a \in B \in D$, and let $\mathscr{D} = \{{}^*R : A \in {}^*R\}$. Suppose ${}^*R_0, \cdots, {}^*R_n \in \mathscr{D}$. Then $D = \bigcap_{k \leqq n} {}^*R_k = {}^*(\bigcap_{k \leqq n} R_k) \in \mathscr{D}$, and therefore, by our earlier work, there exists a single entity $B_D$ satisfying $a \in B_D \in \bigcap_{k \leqq n} {}^*R_k$. Hence $b$ is concurrent on $\mathscr{D}$. This however, by saturation, implies that there exists a set $S$ such that for each $R \in M$, $A \in {}^*R$ implies $a \in S \in {}^*R$. Thus $a \in S \in \mu_d(\{A\})$ which again implies $a \in \cup\, \mu_d(\{A\})$.

We do not know if saturation is necessary even in the case that $A$ has only two elements.

## References

1. W. A. J. Luxemburg, A general theory of monads, in: *Applications of Model Theory to Algebra, Analysis, and Probability*, ed. W. A. J. Luxemburg, Holt, Rinehart and Winston, 1969, pp. 18–86.

Case Western Reserve University
Cleveland, Ohio